# PARAMETERIZED STATIONARY SOLUTION FOR FIRST ORDER PDE

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We analyze the existence of a parameterized stationary solution $z(\lambda, z_0) = \big(x(\lambda, z_0), p(\lambda, z_0),\, u(\lambda, z_0)\big) \in D \subseteq \mathbb{R}^{2n+1}$, $\lambda \in B(0, a) \subseteq \prod_{i=1}^{m}[-a_i, a_i]$, associated with a nonlinear first order PDE, $H_0(x, p(x), u(x)) =$ constant ($p(x) = \partial_x u(x)$) relying on
(a) first integral $H \in \mathcal{C}^\infty\big(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}\big)$ and the corresponding Lie algebra of characteristic fields is of the finite type;
(b) gradient system in a Lie algebra finitely generated over orbits $(f.g.o; z_0)$ starting from $z_0 \in D$ and their nonsingular algebraic representation.

*Keywords*: Gradient system in a $(f.g.o; z_0)$ Lie algebra, Lie algebra of characteristic fields

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## 1. INTRODUCTION

Let $H_0(x, p, u)$, $z = (x, p, u) \in B(z_0.2\rho) \subseteq \mathbb{R}^{2n+1}$, be a second order continuously differentiable function, $H_0 \in \mathcal{C}^2\big(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}\big)$ and consider the equation

$$H_0(z) = H_0(z_0) \text{ for } z = (x, p, u) \in D \subseteq \mathbb{R}^{2n+1},\ z_0 \in D\,. \tag{1}$$

In other words find a manifold $D \subseteq B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}$ such that the equation (1) is satisfied for any $z \in D$. In the case that manifold $D \subseteq \mathbb{R}^{2n+1}$ can be described as follows

$$D = \big\{\big(x, p(x), u(x)\big) \in \mathbb{R}^{2n+1} :\ p(x) = \partial_x u(x), x \in B(x_0, \rho \subseteq \mathbb{R}^n)\big\}, \tag{2}$$

we call it as the standard stationary solution associated with the nonlinear first order PDE given in (1).

It relies on the flow $\{\widehat{z}(t, \lambda) \in \mathbb{R}^{2n+1} : \widehat{z}(0, \lambda) = \widehat{z}_0(\lambda),\, t \in (-a, a),\, \lambda \in \Lambda \subseteq \mathbb{R}^{n-1}\}$ generated by the characteristic field $Z_0(z) \overset{\text{def}}{=} \big(\partial_p H_0(z), P_0(z), <p, \partial_p H_0(z)>\big)$, $P_0(z) = -[\partial_x H_0(z) + p\partial_u H_0(z)]$, of the smooth scalar function $H_0 \in \mathcal{C}^2(\mathbb{R}^{2n+1})$. Using that $H_0$ is a first integral for the characteristic field $Z_0$, we get

$$H_0\big(\widehat{z}(t, \lambda)\big) = H_0\big(\widehat{z}_0(\lambda)\big)\,,\ t \in (-a, a),\, \lambda \in \Lambda \subseteq \mathbb{R}^{n-1}\,, \tag{3}$$

for any parameterized Cauchy conditions

$$\widehat{z}_0(\lambda) = \big(\widehat{x}_0(\lambda), \widehat{p}_0(\lambda), \widehat{u}_0(\lambda)\big)\,, \tag{4}$$

fulfilling the compatibility condition

(5)

$$\partial_{\lambda_i}\widehat{u}_0(\lambda) = \left\langle p_0(\lambda), \partial_{\lambda_i}\widehat{x}_0(\lambda)\right\rangle,\ i = 1,\dots,n-1,\ \lambda = (\lambda_1,\dots,\lambda_{n-1}) \in \Lambda \subseteq \mathbb{R}^{n-1}.$$

The standard stationary solution for (1) can be obtained imposing the following new constraints

$$H_0\big(\widehat{z}_0(\lambda)\big) = H_0(z_0),\ \ \lambda \in \Lambda \subseteq \mathbb{R}^{n-1}, \tag{6}$$

and

$$\text{the } n \text{ vectors in } \mathbb{R}^n, \{\partial_p H_0\big(\widehat{z}_0(\lambda)\big), \partial_{\lambda_1}\widehat{x}_0(\lambda), \dots, \partial_{\lambda_{n-1}}\widehat{x}_0(\lambda)\} \subseteq \mathbb{R}^n \tag{7}$$

are linearly independent for any $\lambda \in \Lambda \subseteq \mathbb{R}^{n-1}$. One may notice that the last conditions need to take into consideration very special $H_0 \in \mathcal{C}^2(\mathbb{R}^{2n+1})$ and Cauchy conditions $\{\widehat{z}_0(\lambda) : \lambda \in \Lambda \subseteq \mathbb{R}^{n-1}\}$ such that (7) is fulfilled. Here we propose to construct a parameterized version of Cauchy conditions such that

$$\widehat{z}(\lambda; z_0) = \big(\widehat{x}(\lambda; z_0), \widehat{p}(\lambda; z_0), \widehat{u}(\lambda; z_0)\big) \tag{8}$$

satisfies stationary conditions

$$H_0\big(\widehat{z}(\lambda; z_0)\big) = H_0(z_0),\ \lambda \in \Lambda = \prod_{i=1}^{m}[-a_i, a_i],\ \widehat{z}(0, z_0) = z_0. \tag{9}$$

In addition, the solution in (8) satisfying stationarity conditions (9), will be obtained as a finite composition of flows starting from $z_0 \in \mathbb{R}^{2n+1}$ (orbit of the origin $z_0 \in \mathbb{R}^{2n+1}$) generated by some characteristic fields including $Z_0$.

In the case that the nonsingularity conditions (7) are fulfilled $(m = n)$ then the parameterized version lead us to a standard stationary solution. A first order continuously differentiable $\widehat{z}(\lambda; z_0) : \Lambda \subseteq \mathbb{R}^m \to \mathbb{R}^{2n+1}$ satisfying stationarity conditions (9) will be called a parameterized stationary solution for PDE (1).

For solving nonlinear equation (9), we use the following procedure. First, the nonlinear equations (9) is transformed into a first order linear system of PDE where the unknowns are the characteristic vector fields generated by a finite set of first integrals $\{H_1(z), \dots, H_m(z) : z \in B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}\}$ corresponding to $Z_0$. Then look for a solution of (9) as an orbit in the Lie algebra of characteristic fields generated by $\{Z_1(z), \dots, Z_m(z) : z \in B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}\}$ corresponding to the first integrals $\{H_1(z), \dots, H_m(z) : z \in B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}\}$.

In the particular case when PDE (1) is determined by a function $H_0(x, p)$, the construction of a parameterized stationary solution is analyzed in Theorem 3.1 of section 3. For the general case, the result is given in Theorem 3.2 of section 3.

In the section 2 are included all definitions and some auxiliary results necessary for the main results given in section 3.

The method of using finite composition of flows (orbit) and the corresponding gradient system in a Lie algebra of vector fields has much in common with the references included here (see [1], [2] and [3]) where both parabolic equations with stochastic perturbations and overdetermined system of first order PDE are studied.

This paper is intended to be a new application of the geometric-algebraic methods presented in [3].

## 2. Definitions, formulation of problems and some auxiliary results

Denote $\mathcal{H} = \mathcal{C}^\infty(\mathbb{R}^{2n+1}, \mathbb{R})$ the space consisting of the scalar functions $H(x,p,u) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ which are continuously differentiable of any order. For each pair $H_1, H_2 \in \mathcal{H}$, define the Poisson bracket

$$\{H_1, H_2\}(z) = \big\langle \partial_z H_2(z), Z_1(z) \big\rangle,\ z = (x,p,u) \in \mathbb{R}^{2n+1} \tag{10}$$

where $\partial_z H_2(z)$ stands for the gradient of a scalar function $H_2 \in \mathcal{H}$ and $Z_1(z) = \big(X_1(z), P_1(z), U_1(z)\big) \in \mathbb{R}^{2n+1}$, $z \in \mathbb{R}^{2n+1}$ is the characteristic field corresponding to $H_1 \in \mathcal{H}$. We recall that $Z_1$ is obtained from $H_1 \in \mathcal{H}$ such that the following equation

$$X(z) = \partial_p H_1(z),\ P_1(z) = -[\partial_x H_1(z) + p\partial_u H_1(z)],\ U_1(z) = \big\langle p, H_1(z) \big\rangle \tag{11}$$

are satisfied. The linear mapping connecting an arbitrary $H \in \mathcal{H}$ and its characteristic field can be represented by

$$Z_H(z) = T(p)\big(\partial_z H\big)(z)\,,\ z \in \mathbb{R}^{2n+1}\,. \tag{12}$$

Here the real $(2n+1) \times (2n+1)$ matrix $T(p)$ is defined by

$$T(p) = \begin{pmatrix} O & I_n & \theta \\ -I_n & O & -p \\ \theta^* & p^* & O \end{pmatrix} \tag{13}$$

considering $O =$ zero marrix of $M_{n,m}$, $I_n-$ unity matrix of $M_{n,m}$ and $\theta \in \mathbb{R}^n$ is the null column vector.

We notice that $T(p)$ is a skew symmetric

$$[T(p)]^* = -T(p) \tag{14}$$

and as a consequence, the Poisson bracket satisfies a skew symmetric property

$$\begin{aligned} \{H_1, H_2\}(z) &= \big\langle \partial_z H_2(z), Z_1(z) \big\rangle = \big\langle \partial_z H_2(z), T(p)(\partial_z H_1)(z) \big\rangle \\ &= \big\langle [T(p)]^*(\partial_z H_2)(z), \partial_z H_1(z) \big\rangle = -\{H_2, H_1\}(z) \end{aligned} \tag{15}$$

In addition, the linear space of characteristic fields $K \subseteq \mathcal{C}^\infty(\mathbb{R}^{2n+1}; \mathbb{R}^{2n+1})$ is the image of a linear mapping $S : D\mathcal{H} \to K$ , where $D\mathcal{H} = \{\partial_z H = H \in \mathcal{H}\}$. Using (12), we define

$$S(\partial_z H)(z) = T(p)(\partial_z H)(z)\,,\ z \in \mathbb{R}^{2n+1} \tag{16}$$

where the matrix $T(p)$ is given in (13).

The linear space of characteristic fields $K = S(D\mathcal{H})$ is extended to a Lie algebra $L_k \subseteq \mathcal{C}^\infty(\mathbb{R}^{2n+1}; \mathbb{R}^{2n+1})$, using the standard Lie bracket of vector fields

$$[Z_1, Z_2] = [\partial_z Z_2] Z_1 - [\partial_z Z_1] Z_2\,, Z_i \in K,\ i = 1, 2\,. \tag{17}$$

On the other hand, each $H \in \mathcal{H}$ is associated with a linear mapping

$$\overrightarrow{H}(\varphi)(z) = \{H, \varphi\}(z) = \big\langle \partial_z \varphi(z), Z_H(z) \big\rangle\,,\ z \in \mathbb{R}^{2n+1}\,, \tag{18}$$

for each $\varphi \in \mathcal{H}$, where $Z_H \in K$ is the characteristic vector field corresponding to $H \in \mathcal{H}$ obtained from $\partial_z H$ by $Z_H(z) = T(p)(\partial_z H)(z)$ see (12).

Define a linear space consisting of linear mappings

$$\overrightarrow{\mathcal{H}} = \{\overrightarrow{H} \,:\, H \in \mathcal{H}\} \tag{19}$$

and extend $\overrightarrow{\mathcal{H}}$ to a Lie algebra $L_H$ using the Lie bracket of linear mappings

$$[\overrightarrow{H}_1, \overrightarrow{H}_2] = \overrightarrow{H}_1 \circ \overrightarrow{H}_2 - \overrightarrow{H}_1 \circ \overrightarrow{H}_1 \,. \tag{20}$$

The link between the two lie algebras $L_K$ (extending $K$) and $L_H$ (extending $\overrightarrow{\mathcal{H}}$) is given by a homomorphism of Lie algebras

$$A : L_H \to L_K \,, \;\; A(\overrightarrow{\mathcal{H}}) = K \tag{21}$$

satisfying

$$A\big([\overrightarrow{H}_1, \overrightarrow{H}_2]\big) = [Z_1, Z_2] \in L_K \,, \text{ where } Z_i = A(\overrightarrow{H}_i),\, i = 1, 2 \,. \tag{22}$$

***Remark* 2.1.** *In general, the lie algebra $L_H \supseteq \overrightarrow{\mathcal{H}}$ does not coincide with the linear space $\overrightarrow{\mathcal{H}}$ and as a consequence, we get $K \subseteq L_K$, $L_K \neq K$. It relies upon the fact that the linear mapping $\{\overrightarrow{H_1, H_2}\}$ generated by the Poisson bracket $\{H_1, H_2\} \in \mathcal{H}$ does not coincide with the Lie bracket $[\overrightarrow{H}_1, \overrightarrow{H}_2]$ defined in* (20)*.*

***Remark* 2.2.** *In the particular case when $H_0$ in* (1) *is replaced by a second order continuously differentiable function $H_0(x,p) : \mathbb{R}^{2n} \to \mathbb{R}$ then the above given analysis will be restricted to the space $\mathcal{H} = \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{R})$. If it is the case then the corresponding linear mapping $S : D\mathcal{H} \to K$ is determined by a simplectic matrix*

$$\widehat{T} = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} \,, \;\; D\mathcal{H} = \{\partial_z H :\, H \in \mathcal{C}^\infty(\mathbb{R}^{2n}; \mathbb{R})\} \,. \tag{23}$$

*In addition, the linear spaces $\overrightarrow{\mathcal{H}}$ and $K \subseteq \mathcal{C}^\infty(\mathbb{R}^{2n}; \mathbb{R}^{2n})$ coincide with their Lie algebras $L_H$ and correspondingly $L_K$. It follows from a direct computation and we get*

$$[Z_1, Z_2](z) = \widehat{T}\partial_z H_{12} \,, \;\; z \in \mathbb{R}^n \tag{24}$$

*where $Z_i = T\partial_z H_i \,, i = 1, 2$ and*

$$H_{12} = (z) = \{H_1, H_2\}(z) = \big\langle \partial_z H_2(z), Z_1(z) \big\rangle \tag{25}$$

*in the Poisson bracket associated with two scalar functions $H_1, H_2 \in \mathcal{H}$. Compute*

$$\begin{aligned} \widehat{T}H_{12}(z) &= \widehat{T}[\partial_z^2 H_2(z)]Z_1(z) + \widehat{T}[\partial_z Z_1^*(z)]\partial_z H_2(z) \\ &= [\partial_z Z_2(z)]Z_1(z) + \widehat{T}[(\partial_z H_1)^*(z)\widehat{T}^*]\partial_z H_2(z) \\ &= \partial_z Z_2(z) Z_1(z) - \widehat{T}(\partial_z^2 H_1)(\widehat{T}\partial_z H_2(z)) \\ &= [\partial_z Z_2(z)]Z_1(z) - [\partial_z Z_1(z)]Z_2(z) = [Z_1, Z_2](z) \end{aligned} \tag{26}$$

*and the conclusion $\{L_H = \overrightarrow{\mathcal{H}}\,,\; K_k = K\}$ is proved. For the general case, the conclusion of Remark 2.2 is not any more true and the manifold structure involved in the solution of PDE* (1) *will be obtained, using first integrals $\{H_1(z), \ldots, H_m(z) :\, z \in B(z_0, 2\rho)\}\,(m \geq 1)$ corresponding to the fixed characteristic field $Z_0$.*

Denote by $K_0 \subseteq \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n+1})$ the linear space consisting of all characteristic fields $Z \in \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n+1})$

$$Z(z) = T(p)\partial_z H(z)\,, \text{ where } \{H(z) : z \in B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}\} \tag{27}$$

is a first integral for the fixed characteristic vector field $\{Z_0(z) : z \in B(z_0, 2\rho)\}$. Let $L_0 \subseteq \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n+1})$ be the Lie algebra determined by the linear space $K_0 \subseteq L_0$.

***Definition 2.3.*** *We say that $L_0$ is of the finite type over $\mathcal{C}^\infty(B(z_0, 2\rho))$(or $\mathbb{R}$) with respect to $K_0$ if there exists a system of vector fields $\{Z_1(z), \dots, Z_m(z) : z \in B(z_0, 2\rho)\} \subseteq K_0$ such that any Lie bracket $[Z_i, Z_j](z) = \sum_{k=1}^{m} \alpha_{ij}^k(z) Z_k(z),\ z \in B(z_0, 2\rho)$, where $\alpha_{ij}^k \in \mathcal{C}^\infty(B(z_0, 2\rho))$( or $\alpha_{ij}^k \in \mathbb{R}$), $i, j, k \in \{1, \dots, m\}$; $\{Z_1, \dots, Z_m\} \subseteq \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n+1})$ will be called a system of generators for $L_0$.*

***Remark 2.4.*** *In the case that PDE (1) is determined by a scalar function $H_0(x, p) : B(z_0, 2\rho) \subseteq \mathbb{R}^{2n} \to \mathbb{R}$ then the linear space $\widehat{K}_0 \subseteq \mathcal{C}(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n}; \mathbb{R}^{2n})$ is consisting of all characteristic fields*

$$\widehat{Z}(z) = \widehat{T}\partial_z H(z) \tag{28}$$

*where $H(z)$, $z \in B(z_0, 2\rho)$, is first integral of $\widehat{Z}_0(z) = \widehat{T}\partial_z H_0(z)$ (see Remark 2.2). We get that the Lie algebra $\widehat{L}_0$ determined by $\widehat{K}_0$ coincides with $\widehat{K}_0$, $\widehat{L}_0 = \widehat{K}_0$ (see Remark 2.2).*

***Definition 2.5.*** *We say that $\widehat{L}_0$ is of the finite type over $\mathcal{C}^\infty(B(z_0, 2\rho))$ (or $\mathbb{R}$) if there exists a system of vector fields $\{Z_1(z), \dots, Z_m(z) : z \in B(z_0, 2\rho) \subseteq \mathbb{R}^{2n}\} \subseteq \widehat{K}_0$ such that any Lie bracket $[Z_i, Z_j](z) = \sum_{k=1}^{m} \alpha_{ij}^k(z) Z_k(z)$, for $\alpha_{ij}^k \in \mathcal{C}^\infty(B(z_0, 2\rho))$( or $\alpha_{ij}^k \in \mathbb{R}$), $i, j, k \in \{1, \dots, m\}$; $\{Z_1, \dots, Z_m\}$ is called a system of generators for $\widehat{L}_0$.*

***Remark 2.6.*** *The simplest case in our analysis is obtained when PDE (1) is determined by a linear function with respect to $p \in \mathbb{R}^n$, i.e*

$$H_0(x, p) = \langle p, f_0(x) \rangle\,,\ p \in \mathbb{R}^n,\ x \in \mathbb{R}^n,\ z = (x, p),\ z_0 = (x_0, p_0) \tag{29}$$

*where $f_0 \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$. In this case the linear space $\widehat{K}_0$ (see Remark 2.4) coincides with the Lie algebra $\widehat{L}_0$ generated by $\widehat{K}_0$ where*

$$\widehat{K}_0 = \{\widehat{T}\partial_z H(z) : H(z) = \langle p, f(x) \rangle,\ [f, f_0](x) = 0\,, x \in B(z_0, 2\rho) \subseteq \mathbb{R}^n\}. \tag{30}$$

**LEMMA 2.7.** *Assume that PDE (1) is determined by $H_0(x, p) = \langle p, f(x) \rangle$, $x \in B(z_0, 2\rho) \subseteq \mathbb{R}^n$, $f_0 \in \mathcal{C}^2(B(z_0, 2\rho); \mathbb{R}^n)$. Assume that there exists $\{f_1, \dots, f_m\} \subseteq \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^n)$ satisfying*

$$[f_i, f_j](x) = \sum_{k=1}^{m} \alpha_{ij}^k f_k(x),\ x \in B(z_0, 2\rho),\ \text{where } \alpha_{ij}^k \subseteq \mathbb{R},\ k, i, j \in \{1, \dots, m\} \tag{31}$$

$$[f_0, f_i](x) = 0,\ x \in B(z_0, 2\rho),\ i \in \{1, \dots, m\}\,. \tag{32}$$

*Then the lie algebra $\widehat{L}_0$ generated by $\widehat{K}_0$ (see (30)) fulfils*

$$\widehat{L}_0 = \widehat{K}_0 \text{ and } \widehat{L}_0 \text{ is finite dimensional with } dim\widehat{L}_0 \leq m. \tag{33}$$

*Proof.* Define $H_i(z) = \langle p, f_i(x)\rangle$ and the corresponding characteristic field

$$Z_i(z) = \begin{pmatrix} f_i(x) \\ A_i(x)p \end{pmatrix},\ A_i(x) = -[\partial_x f_i(x)]^*,\ 1 \leq i \leq m. \tag{34}$$

Compute a Lie bracket

$$[Z_i, Z_j](z) = \begin{pmatrix} [f_i, f_j](x) \\ P_{ij}(z) \end{pmatrix},\ i, j \in \{1, \ldots, m\},\ x \in B(z_0, 2\rho), \tag{35}$$

where $[f_i, f_j]$ is the Lie bracket using $f_i, f_j \in \mathcal{C}^\infty(B(z_0, 2\rho), \mathbb{R}^n)$. Here $P_{ij}(z)$ is computed as follows

(36)
$$\begin{aligned} P_{ij}(z) &= [\partial_x(A_j(x)p)]f_i(x) + A_j(x)A_i(x)p - [\partial_x A_i(x)p]f_j(x) \\ &= -A_i(x)A_j(x)p = \partial_x\big[ < A_j(x)p, f_i(x) > - < A_i(x)p, f_j(x) > \big] \\ &= \partial_x\langle p, [f_i, f_j](x)\rangle \end{aligned}$$

where $P_{ij}(z) \stackrel{\text{def}}{=} \big[\partial_z(A_j(x)p)\big]Z_i(z) - \big[\partial_z(A_i(x)p)\big]Z_j(z),\ x \in B(x_0, 2\rho)$, is used. It shows (see (24)) that the Lie bracket $[Z_i, Z_j](z)$ in (35) can be written as a characteristic vector field associated to $H_{ij}(z) \stackrel{\text{def}}{=} \langle p, [f_i, f_j](x)\rangle$, $x \in B(z_0, 2\rho) \subseteq \mathbb{R}^n$. As a consequence, assuming that $\{f_1, \ldots, f_m\}$ satisfies (31) and (32), we get that $\{Z_1, \ldots, Z_m\} \subseteq \widehat{K}_0$ is a system of generators for $\widehat{K}_0$. The proof is complete. □

**LEMMA 2.8.** *Assume that PDE (1) is determined by a second order continuously differentiable $H_0(x,p) : B(z_0, 2\rho) \to \mathbb{R}$, and the Lie algebra $\widehat{L}_0 = \widehat{K}_0$ defined in Remark 2.4 is of the finite type over $\mathbb{R}$. Then there exists a parameterized stationary solution of PDE (1) given by*

$$\widetilde{z}(\lambda, z_0) = G_1(t_1) \circ \ldots \circ G_m(t_m)[z_0],\ \lambda = (t_1, \ldots, t_m) \in \prod_{i=1}^{m}[-a_i, a_i] = \Lambda, \tag{37}$$

*where $\{G_i(\sigma)[y] :\ \sigma \in [-a_1, a_i],\ y \in B(z_0, \rho)\}$ is the local flow generated by the vector field $Z_i \in \mathcal{C}^\infty(B(z_0, 2\rho), \mathbb{R}^{2n})$ and $\{Z_1, \ldots, Z_m\} \subseteq \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n})$ is a system of generators for $\widehat{L}_0$.*

*Proof.* Assuming that $\widehat{L}_0$ is of the finite type over $\mathbb{R}$, we notice that fixing a system of generators $\{Z_1(z), \ldots, Z_m(z) : z \in B(z_0, 2\rho) \subseteq \mathbb{R}^{2n}\} \subseteq \widehat{K}_0$ for $\widehat{L}_0$ we may and do construct a finite dimensional Lie algebra $L(Z_1, \ldots, Z_m) \subseteq \widehat{L}_0$ for which $\{Z_1, \ldots, Z_m\} \subseteq \widehat{K}_0$ is a system of generators over $\mathbb{R}$. It allows (see [3]) to define a corresponding gradient system in the finite dimensional Lie algebra $L(Z_1, \ldots, Z_m)$ associated with the following composition of local flows

$$\widehat{z}(\lambda, z_0) = G_1(t_1) \circ \ldots \circ G_m(t_m)[z_0],\ \lambda = (t_1, \ldots, t_m) \in \prod_{i=1}^{m}[-a_i, a_i] = \Lambda. \tag{38}$$

Here $\{G_i(\sigma)[y] :\ \sigma \in [-a_1, a_i],\ y \in B(z_0, \rho)\}$ is the local flow generated by the vector field $Z_i \in \widehat{K}_0$ and $\{Z_1, \ldots, Z_m\}$ is the system of generators. In addition, there exists analytic vector fields $\{q_1(\lambda), \ldots, q_m(\lambda) :\ \lambda \in B(0, a) \subseteq$

$\Lambda\} \subseteq \mathcal{C}^{\omega}(B(0,a);\mathbb{R}^m)$ such that each vector field $Z_i\big(\widehat{z}(\lambda, z_0)\big)$, $\lambda \in B(0,a)$, can be recovered by taking the Lie derivative

$$\partial_\lambda \widehat{Z}(\lambda; z_0) q_i(\lambda) = Z_i\big(\widehat{z}(\lambda . z_0)\big),\ \lambda \in B(0,a) \subseteq \Lambda,\ \ i \in \{1,\dots,m\} \tag{39}$$

and

$$\{q_1(\lambda),\dots,q_m(\lambda)\} \subseteq \mathbb{R}^m \text{ are linearly independent for any } \lambda \in B(0,a) \subseteq \Lambda. \tag{40}$$

The manifold defined in (38) stands for the parameterized stationary solution of PDE (1) and by a direct computation, we get

$$\big\langle \partial_\lambda H_0\big(\widehat{z}(\lambda, z_0)\big), q_i(\lambda)\big\rangle = \big\langle \partial_z H_0\big(\widehat{z}(\lambda, z_0)\big), Z_i\big(\widehat{z}(\lambda, z_0)\big)\big\rangle = 0 \tag{41}$$

for any $\lambda \in B(0,a) \subseteq \Lambda$ and $i \in \{1,\dots,m\}$. Using (40), we notice that (41) lead us to

$$\partial_\lambda H_0\big(\widehat{z}(\lambda, z_0)\big) = 0,\ \forall\, \lambda \in B(0,a) \subseteq \Lambda \tag{42}$$

and the proof is complete. $\square$

## 3. Main results

With the same notations as in section 2 and considering that Lie algebra $\widehat{L}_0 = \widehat{K}_0$, defined in Remark 2.4, is of the finite type over $\mathcal{C}^\infty(B(z_0, 2\rho), \mathbb{R}^{2n})$ (see definition 2.5), we get

**THEOREM 3.1.** *Assume that PDE (1) is determined by $H_0 \in \mathcal{C}^2(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n})$ and the Lie algebra $\widehat{L}_0 = \widehat{K}_0$ is of the finite type over $\mathcal{C}^\infty(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n})$. Then there exists a parameterized stationary solution of PDE (1) given by*

$$\widehat{z}(\lambda, z_0) = G_1(t_1) \circ \ldots \circ G_m(t_m)[z_0],\ \lambda = (t_1,\dots,t_m) \in B(0,a) \subseteq \prod_{i=1}^{m}[-a_i, a_i] = \Lambda. \tag{43}$$

*where $\{G_i(\sigma)[y] :\ \sigma \in [-a_1, a_i],\ y \in B(z_0, \rho)\}$ is the local flow generated by the vector field $Z_i \in \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n})$ and $\{Z_1,\dots,Z_m\} \subseteq \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n})$ is the system of generators for $\widehat{L}_0$.*

*Proof.* By hypothesis, let $\{Z_1,\dots,Z_m\} \subseteq \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n})$ be a system of generators for $\widehat{L}_0$ which is assumes of the finite type over $\mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n})$. Define an orbit of $\widehat{L}_0$ starting from $z_0$.

$$\widetilde{z}(\lambda, z_0) = G_1(t_1) \circ \ldots \circ G_m(t_m)[z_0],\ \lambda = (t_1,\dots,t_m) \in \prod_{i=1}^{m}[-a_i, a_i] = \Lambda. \tag{44}$$

where $\{G_i(\sigma)[y] :\ \sigma \in [-a_1, a_i],\ y \in B(z_0, \rho)\}$ is the local flow generated by the vector field $Z_i \in \{Z_1,\dots,Z_m\}$, $i \in \{1,\dots,m\}$. Let $L(Z_1,\dots,Z_m) \subseteq \mathcal{C}^\infty(B(z_0, 2\rho); \mathbb{R}^{2n})$ be the Lie algebra generated by $\{Z_1,\dots,Z_m\}$ and notice that $L = L(Z_1,\dots,Z_m)$ is finitely generated over orbits starting from $z_0$, which is abbreviated as $(f.g.o; z_0)$ in [3]. On the other hand, using the orbit starting

from $z_0$ defined in (44), we associate a gradient system in $L(Z_1, \ldots, Z_m)$.

(45)

$$\begin{aligned}\partial_1 \widehat{z}(\lambda, z_0) &= Z_1\big(\widehat{z}(\lambda, z_0)\big),\ \partial_2 \widehat{z}(\lambda, z_0) = X_2\big(t_1; \widehat{z}(\lambda, z_0)\big), \ldots, \\ &= \partial_1 \widehat{z}(\lambda, z_0) = X_m\big(t_1, \ldots, t_{m-1}; \widehat{z}(\lambda, z_0)\big),\ \lambda = (t_1, \ldots, t_m) \in\end{aligned}$$

$$\prod_{i=1}^{m} [-a_i, a_i] = \Lambda$$

where $\partial_i \widehat{z}(\lambda, z_0) \overset{\text{def}}{=} \partial_{t_i} \widehat{z}(\lambda, z_0),\ i \in \{1, \ldots, m\}$.

Using the algebraic representation of a gradient system determined by a system of generators $\{Z_1, \ldots, Z_m\} \subset L$ in a $(f.g.o; z_0)$ Lie algebra $L$ (see [3]), we get

$$\partial_\lambda \widehat{z}(\lambda, z_0) = \{Z_1, \ldots, Z_m\}(\widehat{z}(\lambda, z_0))A(\lambda),\ \lambda \in \Lambda,\ A(0) = I_m, \tag{46}$$

where the $(m \times m)$ matrix $A(\lambda)$ is nonsingular for any $\lambda \in B(0, a) \subseteq \Lambda$ with some $a > 0$. It lead us to get $\big\{\{q_1(\lambda), \ldots, q_m(\lambda)\} \subseteq \mathbb{R}^m : \lambda \in B(0, a)\big\}$ such that

(47)

$$\{q_1, \ldots, q_m\} \subseteq \mathcal{C}^\infty(B(0, a); \mathbb{R}^m) \text{ are linearly independent } \forall\, \lambda \in B(0, a),$$

$$\partial_\lambda \widehat{z}(\lambda, z_0) q_i(\lambda) = Z_i\big(B(0, a); \mathbb{R}^m\big),\ \lambda \in B(0, a) \subseteq \Lambda,\ i \in \{1, \ldots, m\}. \tag{48}$$

The equation (48) allows us to get the conclusion

$$\partial_\lambda \widehat{z}(\lambda, z_0) q_i(\lambda) = 0,\ \lambda \in B(0, a) \subseteq \Lambda,\ i \in \{1, \ldots, m\}. \tag{49}$$

and using (47), we obtain $\partial_\lambda H_0(\widehat{z}(\lambda, z_0)) = 0,\ \lambda \in B(0, a) \subseteq \Lambda$ and the proof is complete. □

With the same notations as in section 2 and considering that the Lie algebra $L_0$ is of the finite type over $\mathcal{C}^\infty(B(z_0, 2\rho), \mathbb{R}^{2n+1})$ with respect to $K_0 \subseteq L_0$ (see definition 2.3), we get

**THEOREM 3.2.** *Assume that PDE* (1) *is determined by* $H_0 \in \mathcal{C}^\infty(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1})$ *and* $L_0 \supset K_0$ *is of the finite type over* $\mathcal{C}^\infty(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1})$ *with respect to* $K_0$. *Let* $\{Z_1, \ldots, Z_m\} \subseteq \mathcal{C}^\infty(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}, \mathbb{R}^{2n+1}) \cap K_0$ *be a system of generators for* $L_0$. *Then there exists a parameterized stationary solution for PDE* (1) *given by*

(50)

$$\widehat{z}(\lambda, z_0) = G_1(t_1) \circ \ldots \circ G_m(t_m)[z_0],\ \lambda = (t_1, \ldots, t_m) \in B(0, a) \subseteq \prod_{i=1}^{m} [-a_i, a_i] = \Lambda.$$

*where* $\{G_i(\sigma)[y] :\ \sigma \in [-a_1, a_i],\ y \in B(z_0, \rho)\}$ *is the local flow generated by the vector field* $Z_i$, $1 \leq i \leq m$.

*Proof.* By hypothesis, any Lie bracket $[Z_i, Z_j](z)$ is a linear combination of $\{Z_1, \ldots, Z_m\}(z)$, $z \in B(z_0, 2\rho)$, using smooth functions from $\mathcal{C}^\infty(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1})$. It shows that the Lie algebra $L = L(Z_1, \ldots, L_m) \subseteq \mathcal{C}^\infty(B(z_0, 2\rho) \subseteq \mathbb{R}^{2n+1}; \mathbb{R}^{2n+1})$ generated by the fixed $\{Z_1, \ldots, Z_m\}$ is finitely generated over orbits starting from $z_0$ (see $(f.g.o; z_0)$ Lie algebra in [3]). In addition, $\{Z_1, \ldots, Z_m\}$ is a system of generators for $L$. Using the orbit defined in (50) we proceed as in proof of Theorem 3.1 (see (45)-(49)) and get the conclusion. The proof is complete. □

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## References

[1] B. Iftimie and C. Vârsan, *A Pathwise Solution for Nonlinear Parabolic Equation with Stochastic Perturbations*. Cent. Eur. J. Math. 3(2003), 1, 367-381.

[2] B. Iftimie and C. Vârsan, *Evolution System of Cauchy-Kowalewska and arabolic Type with Stochastic Perturbations*. Math. Rep. 10(2008), 3, 213-238.

[3] C. Vârsan, *Applications of Lie Algebras to Hyperbolic and Stochastic Differential Equations*. Khuwer Academic Publishers, 1999.

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